ETMAG LECTURE 2

Complex numbers continued

We can rewrite the definition of addition and multiplication of complex numbers in the language of points from the complex plane:

$(a,b)+(c,d) = (a+c,b+d)$ which corresponds to geometrical addition of vectors anchored at $(0,0)$ with endpoints at (a,b) and (c,d), respectively, and

$(a,b)\cdot(c,d) = (ac-bd,ad+bc)$ (geometrical meaning of this operation is more complicated).

Hence, we can look at the algebra of complex numbers as an extension of arithmetic from the set of real numbers to the set of pairs of real numbers.

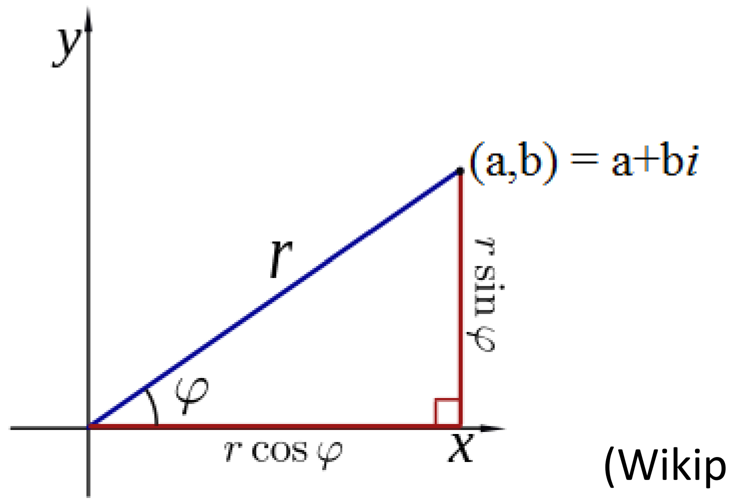
Notice that both complex addition and multiplication, when performed on complex numbers with imaginary parts are just "normal" arithmetic operations:
(a,0)(c,0) = (ac-0·0,a·0+0·c) = (ac,0). Or, in the standard form, (a+0*i*)(c+0*i*) = ac-0·0+(a0+0c)*i* = ac. The same for addition.

The complex modulus, when applied to a real number gives the "normal" absolute value: |a+0*i*| = $\sqrt{a^2+0^2} = \sqrt{a^2}$ = |a|.

Comprehension test

1. The same happens for complex division.
2. $\overline{\mathrm{z}}$ = z if and only if z is a real number
3. $\overline{\overline{\mathrm{z}}} = \mathrm{z}$
4. $\overline{\mathrm{z}+\mathrm{w}} = \overline{\mathrm{z}} + \overline{\mathrm{w}}$
5. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

A point z of the plane can be identified by its Cartesian coordinates, say (a,b), but also by its *polar coordinates*, i.e. the distance r from the origin and the angle φ between positive half-axis OX and the segment (0,0)(a,b). Hence, (a,b) = ($r\cos\varphi$,$r\sin\varphi$) or, equivalently, z = a+bi = r($\cos\varphi$+ $i\sin\varphi$). Clearly, r =$\sqrt{a^2+b^2}$, i.e. r = |z|



(Wikipedia & TT)

Definition.

The formula r($\cos\varphi$+ $i\sin\varphi$) is known as the *polar form* (sometimes *trigonometric form*) of the complex number z.

Remarks.

- (-1)($cos\varphi$+ i$sin\varphi$) is NOT a polar form of a complex number
- 7($cos\ \alpha$ + i$sin\ \varphi$) is NOT a polar form (unless $\alpha = \ \varphi$)
- 666($cos\varphi -$ i$sin\varphi$) is NOT a polar form (unless $sin\ \varphi$ = 0)
- The question "where the hell is this imaginary unit *i*" suddenly becomes meaningful. The answer is "at (0,1)".

Comprehension

1. Find the polar form of 1, -1, *i* and $-i$.
2. Find the polar form of 666($cos\ \alpha -$ i$sin\ \alpha$)
3. Knowing that the polar form of *z* is r($cos\ \alpha$ + i$sin\ \alpha$) find the polar form of $\overline{z}$**.**

The angle φ is called an argument of z. Since both sine and cosine are periodic function with the period of 2π, a complex number has infinitely many arguments, each of the form $\varphi+2k\pi$ for some integer k. Hence the term "THE polar form of z" is a slight abuse of language.

Definition.
The argument of z belonging to the interval $<0;2\pi)$ is called the *principal argument* of z.

Example.
The polar form of $z=1+i$ is $\sqrt{2}(cos\frac{\pi}{4} + i\ sin\frac{\pi}{4})$, the polar form of z=1 is $cos0 + isin0$, for z=-1 is $\cos\pi + i\sin\pi$

It helps if you memorize values of sine and cosine for those basic angles 0, $\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{2}$ and the like.

Theorem (Multiplication Lemma)

Let z = $r(\cos \alpha + i \sin \alpha)$ and $w = t(\cos \varphi + i \sin \varphi)$ be two complex numbers. Then

$$zw = rt(\cos(\alpha+\varphi) + i \sin(\alpha+\varphi)).$$

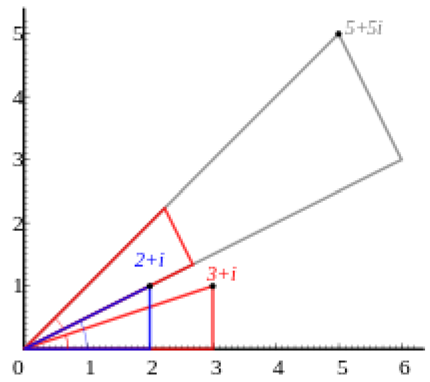
Proof.

$zw = r(\cos \alpha + i \sin \alpha)t(\cos \varphi + i \sin \varphi) = rt((\cos \alpha \cos \varphi - \sin \alpha \sin \varphi) + i(\cos \alpha \sin \varphi + \sin \alpha \cos \varphi)) = rt(\cos(\alpha+\varphi) + i \sin(\alpha+\varphi))$. The last transformation follows from well-known trigonometric identities. QED

Remark.

Another way of representing a complex number $z = r(\cos \alpha + i \sin \alpha)$ is the *exponential form* $z=re^{i\alpha}$. By laws of exponentiation we obtain a similar law: $zw = rt\, e^{i(\alpha+\varphi)}$.

Remark. This is as close as we can get to a geometrical interpretation of complex multiplication: when you multiply *z* by *w* you rotate the vector representing *z* counterclockwise by the argument of *w* and you adjust the length so that it becomes the product of lengths of *z* and *w.*



Picture from Wikipedia

Corollary (of Multiplication Lemma)

Let z = $r(\cos\alpha + i\sin\alpha)$ and $w = t(\cos\varphi + i\sin\varphi)$ be two complex numbers. Then

$$\frac{z}{w} = \frac{r}{t}(\cos(\alpha-\varphi) + i\sin(\alpha-\varphi)).$$

Proof.

$\frac{z}{w}$ is the only number x satisfying $xw = z$.

Try $x = \frac{r}{t}(\cos(\alpha-\varphi) + i\sin(\alpha-\varphi))$.

Using the Multiplication Lemma we obtain

$xw = \frac{r}{t}(\cos(\alpha-\varphi) + i\sin(\alpha-\varphi)) \cdot t(\cos\varphi + i\sin\varphi) =$

$r(\cos((\alpha-\varphi)+\varphi) + i\sin((\alpha-\varphi)+\varphi)) = r(\cos\alpha + i\sin\alpha) =$

z hence, $x = \frac{r}{t}(\cos(\alpha-\varphi) + i\sin(\alpha-\varphi)) = \frac{z}{w}$. QED

Corollary (de Moivre Law)

Let z = $r(\cos\alpha + i\sin\alpha)$. Then for every positive integer n

$$z^n = r^n(\cos n\alpha + i\sin n\alpha).$$

Proof.
The formula follows from a repeated application of the Multiplication Lemma. (Use induction if you want to be VERY rigorous). QED

Remark. This means that when you raise z of modulus 1 to n-th power, geometrically you rotate z counterclockwise n-1 times by α (the modulus stays 1).

Example.

Calculate z^{10} where $z = 1 + i\sqrt{3}$. We will use de Moivre Law. First, we find the modulus of z and factor it out. Since $|z| = 2$ we can write $z = 2(\frac{1}{2} + i\,\frac{\sqrt{3}}{2})$. The number in parenthesis belongs to the unit circle, hence, there exists α such that $\cos\alpha = \frac{1}{2}$ and $sin\alpha = \frac{\sqrt{3}}{2}$. If you recall your high school algebra the angle is $\frac{\pi}{3}$, i.e. z=$2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ and $z^{10} = 2^{10}(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3})$ = 1024 $(\cos(2\pi + \frac{4\pi}{3}) + i\sin(2\pi + \frac{4\pi}{3}))$ = 1024 $(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3}))$ = $1024(-\frac{1}{2} + i(-\frac{\sqrt{3}}{2}))$.

This is much more fun than going $(1 + i\sqrt{3})\,(1 + i\sqrt{3})\,\ldots$ ten times.

Definition 1.2.
Every complex number *w* satisfying the equation w^n=*z* is called a *root of z of order n*.

Notice that, unlike in real numbers, both -2 and 2 are called square roots of 4. De Moivre Law can be used to calculate complex roots.

Root formula
Take *z*=r(cos α+*i*sin α) and suppose *w*=p(cosφ +*i*sin φ) is a root of *z* of order n. Then $w^n = p^n(cos$ nφ + i sin nφ) = r(cos α +isin α). Hence, $p= \sqrt[n]{r}$ (in the usual, real-number sense) and cos nφ =cos α and sin nφ =sin α.

Since 2π is the period of both sine and cosine, we obtain

$n\varphi_k = \alpha + k2\pi$ or, equivalently, $\varphi_k = \frac{\alpha+2k\pi}{n}$, for k=0,1,2, ...

So, $w_k = \sqrt[n]{r}(cos\frac{\alpha+2k\pi}{n} + i\sin\frac{\alpha+2k\pi}{n})$.

Consider two roots whose indices differ by n, say w_k and w_{k+n}.

$$w_{k+n} = \sqrt[n]{r}\left(cos\,\frac{\alpha+2(k+n)\pi}{n} + i\sin\frac{\alpha+2(k+n)\pi}{n}\right) =$$

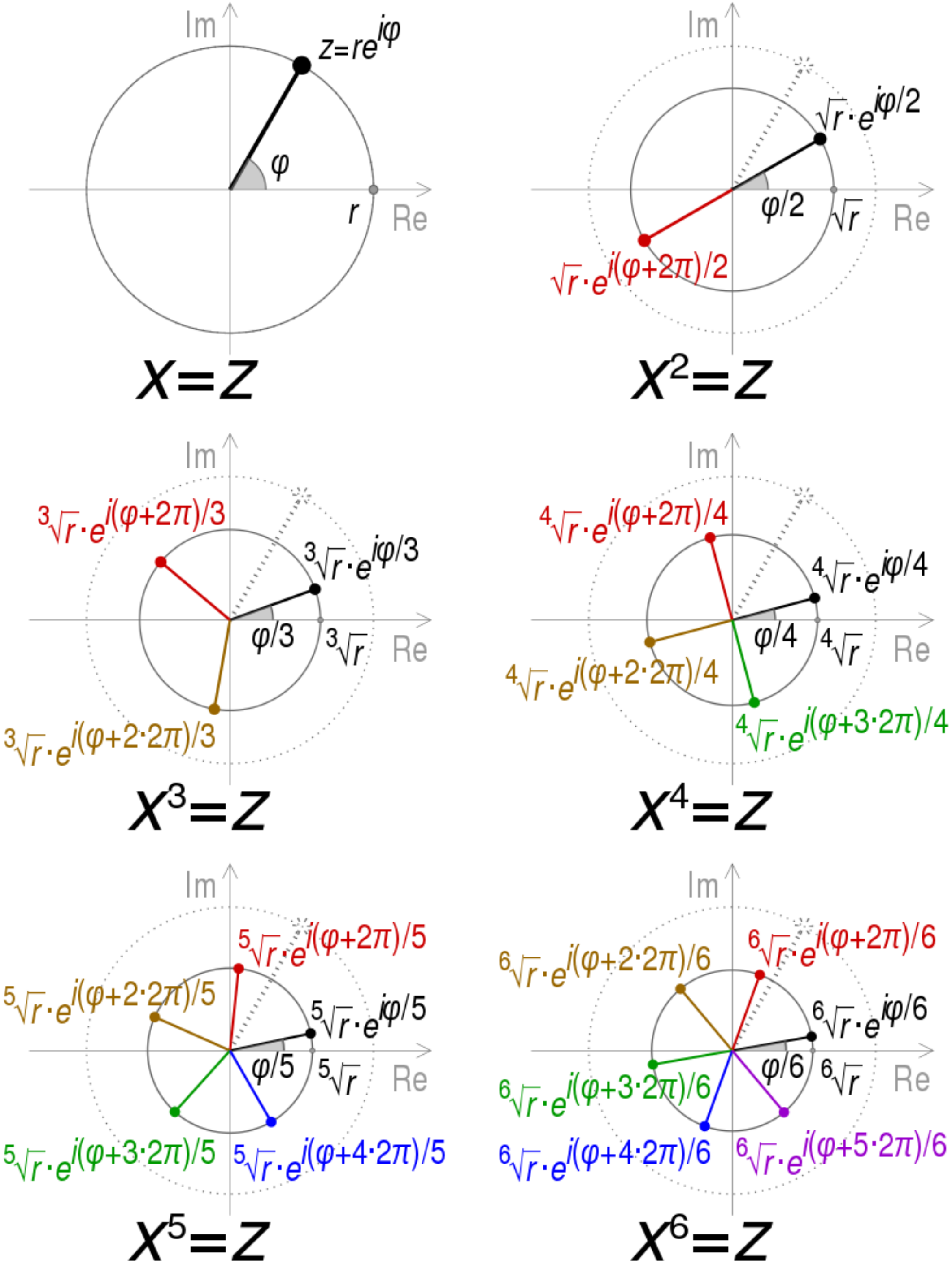
$$\sqrt[n]{r}\left(cos\,\frac{\alpha+2k\pi+2n\pi}{n} + i\sin\frac{\alpha+2k\pi+2n\pi}{n}\right) =$$

$$\sqrt[n]{r}\left(\cos\left(\frac{\alpha+2k\pi}{n} + 2\pi\right) + i\sin\left(\frac{\alpha+2k\pi}{n} + 2\pi\right)\right) =$$

$$\sqrt[n]{r}\left(\cos\left(\frac{\alpha+2k\pi}{n}\right) + i\sin\left(\frac{\alpha+2k\pi}{n}\right)\right) = w_k.$$

This indicates that we only get n different roots of z of order n, namely $w_0, w_1, \ldots, w_{n-1}$ – no more, no less.

Important. The root formula and its consequences apply only to roots of a number **not to roots of a polynomial**.



Picture from Wikipedia.

Roots of order n of a complex number z are uniformly distributed over the circle centered at 0 and with the radius of $\sqrt[n]{r}$. The angular distance between any two consecutive roots is $\frac{2\pi}{n}$.

Polynomials

Definition. A *polynomial of degree n* over a set K is any function f: K→K of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n \in K$ and $a_n \neq 0$. The set of all polynomials over K is denoted by K[x].

We adopt the convention that the degree of the zero polynomial 0 is $= -\infty$. For other constant polynomials, the degree is 0. K will usually be the set $\mathbb{R}$ or $\mathbb{C}$.

Definition.
Let *f* be a polynomial over K. A number *a* from *K* is called a *root* of *f* if and only if $f(a) = 0$.

Fact. (Remainder lemma)
For every two polynomials f, $g \in K[x]$ with $g \neq 0$ there exist polynomials q, $r \in K[x]$ such that

$$f = qg + r \text{ and } deg(r) < deg(g).$$

Corollary.

A number a is a root of a polynomial f(x) if and only if f(x) is divisible by $(x - a)$.

Theorem(Main Theorem of Algebra

Every polynomial $f \in \mathbb{C}[x]$ of degree at least 1 has a root in $\mathbb{C}$.

Corollary.

Every polynomial $f \in \mathbb{C}[x]$ of degree n has exactly n roots (counting multiplicities).

Theorem.
If $f \in \mathbb{R}[x]$ then for every root z of f, $\bar{z}$ is also a root of f.
Proof.
$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$. Hence, $\overline{f(z)} = \bar{0} = 0$, and

$\overline{f(z)} = \overline{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0} =$

$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \dots \overline{a_1 z} + \overline{a_0} =$

$a_n \bar{z}^n + a_{n-1} \bar{z}^{\,n-1} + \dots + a_1 \bar{z} + a_0 = f(\bar{z}) = 0.$

Corollary.
If $f \in \mathbb{R}[x]$ then f can be factored into a product of polynomials from $\mathbb{R}[x]$ of degree at most 2 each.
Proof.
It follows from the fact that $\big(x - (a + bi)\big)\big(x - (a - bi)\big) =$
$x^2 - x(a - bi) - x(a + bi) + (a + bi)(a - bi) =$
$x^2 - 2ax + xbi - xbi + a^2 + b^2.$

Comprehension.

Prove on your own: every polynomial from $\mathbb{R}[x]$ with on odd degree has at least one real root.